Aspects of the ODE/IM correspondence

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3rd South East Mathematical Physics Seminar

Plan

- Introduction to the ODE/IM correspondence
- Generalisations of the correspondence
  - Third-order ODE
  - PDEs and massive quantum field theories

The ODE/IM Correspondence

What is the ODE/IM correspondence?

(Dorey, Dunning, Tateo, Bazhanov, Lukyanov, Zamolodchikov, Suzuki, ...)

⟶ Linear ordinary differential equations (ODEs) defined in the complex plane

- ∘ Spectral determinants related to certain eigenvalue problems for the ODEs.

⟶ Quantum integrable models (IMs)/quantum field theories (QFTs) in two dimensions

- ∘ Baxter's TQ relation, T and Q operators of Bazhanov, Lukyanov, Zamolodchikov for quantum field theory.

- ⋆ Correspondence based mainly on functional relations (and Bethe Ansatz-type equations) that appear on both sides

The ODE/IM Correspondence

- Eigenvalues of ODEs involved satisfy Bethe Ansatz-type equations (BAE)
- Real eigenvalues $\longrightarrow$ mapped to BAE with real solutions: ground state of quantum integrable model
- Some results on excited states [1]

[1] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, Higher-level eigenvalues of Q-operators and Schrödinger equation, Adv Theor and Math Phys 7 (2004), 711-725

The ODE/IM Correspondence

ODEs successfully mapped to Bethe Ansatz systems

$$A_{n-1}: \quad \Big((-1)^{n+1} D_n(\mathbf{g}) + p(x,E)\Big)\psi(x) = 0$$

$$D_n: \quad \left(D_n(\mathbf{g}^\dagger)\left(\frac{d}{dx}\right)^{-1} D_n(\mathbf{g}) - \sqrt{p(x,E)}\left(\frac{d}{dx}\right)\sqrt{p(x,E)}\right)\psi(x) = 0$$

$$B_n: \quad \left(D_n(\mathbf{g}^\dagger) D_n(\mathbf{g}) + \sqrt{p(x,E)}\left(\frac{d}{dx}\right)\sqrt{p(x,E)}\right)\psi(x) = 0$$

$$C_n: \quad \left(D_n(\mathbf{g}^\dagger)\left(\frac{d}{dx}\right) D_n(\mathbf{g}) - p(x,E)\left(\frac{d}{dx}\right)^{-1} p(x,E)\right)\psi(x) = 0$$

- $p(x,E) = x^{nM} - E$
- $D_n(\mathbf{g}) = D\left(g_{n-1} - (n-1)\right) D\left(g_{n-2} - (n-2)\right) \ldots D\left(g_0\right)$, with $D(g) = (d/dx - g/x)$
- $\mathbf{g} = \{g_{n-1}, \ldots, g_1, g_0\}$ and $\mathbf{g}^\dagger = \{n-1-g_0, n-1-g_1, \ldots, n-1-g_{n-1}\}$

The ODE/IM Correspondence

Applications/Connections:

- $\mathcal{PT}$-symmetric Quantum Mechanics
- Quasi-exact solvability of differential equations
- Spectral equivalences between differential equations

  and others...

The ODE/IM Correspondence

Goals:

- Further explore connections
- Enlarge sets of Differential Equations that enter the correspondence
- Generalise to other quantum integrable models

Third-order ODE

One direction of extension:

- Consider the third-order ODE:

$$\left(\frac{d^3}{dx^3}+\frac{A}{x^2}\frac{d}{dx}+\frac{B}{x^3}+x^{3M}+\alpha x^{2M-1}-E\right)\psi(x)=0$$

with $A=-2+g_0g_1+g_0g_2+g_1g_2\,,\quad B=-g_0g_1g_2\quad$ and $\quad g_i,M,\alpha\in\mathbb{R}$

- Find E for which there exists a solution decaying exponentially as $x\to+\infty$ and algebraically as $x\to 0$

Third-order ODE

- The ODE admits the following asymptotic solution

$$y \sim x^{-M-\frac{\alpha}{3}} \exp\left(-\frac{x^{M+1}}{M+1}\right)$$

as $|x| \to \infty$ in the sector $|\arg x| < 4\pi/(3M+3)$

- We define rotated solutions that decay in certain sectors of the complex plane

$$y_k(x, E, \alpha, \mathbf{g}) \equiv \omega^{k-\frac{\alpha k}{3}} \omega^{-k(M+1)} y(\omega^{-k} x, \omega^{-3kM} E, \omega^{-k(M+1)} \alpha, \mathbf{g})$$

with $\omega = \exp\left(\frac{2\pi i}{3(M+1)}\right)$

- For $k \in \mathbb{N}$ the y_k solve the ODE. Also define

$$\alpha_0 \equiv \alpha\,, \quad \alpha_1 \equiv \omega^{-(M+1)}\alpha\,, \quad \alpha_2 \equiv \omega^{-2(M+1)}\alpha$$

Third-order ODE

- The functions y_k, y_{k+1}, y_{k+2} are linearly independent

$$y = C^{(1)}(E,\alpha)\, y_1 + C^{(2)}(E,\alpha)\, y_2 + C^{(3)}(E,\alpha)\, y_3$$

  The $C^{(i)}$ are called Stokes multipliers and can be expressed in terms of Wronskians of rotated solutions y_k

- Expand the solution y in basis of solutions to the ODE at the origin

$$y = D_0(E,\alpha_i)\chi_0 + D_1(E,\alpha_i)\chi_1 + D_2(E,\alpha_i)\chi_2$$

  with $\chi_i \sim x^{g_i}$, $i = 1, 2, 3$

Third-order ODE

- Combining the relations for solutions at the origin and at infinity

$$
\begin{aligned}
C^{(1,i)}(E)D^{(1,i+1)}(\omega^{-3M}E)D^{(2,i+1)}(\omega^{-3M}E) = \\
\omega^{g_0-1+\frac{\alpha_{i+1}}{3}}D^{(1,i)}(E)D^{(2,i+1)}(\omega^{-3M}E) \\
+\,\omega^{g_1-1+\frac{1}{3}(2\alpha_{i+1}-\alpha_{i+2})}D^{(1,i+2)}(\omega^{-6M}E)D^{(2,i)}(E) \\
+\,\omega^{2-g_0-g_1-\frac{1}{3}(2\alpha_i+\alpha_{i+2}-\alpha_{i+1})}D^{(1,i+1)}(\omega^{-3M}E)D^{(2,i+2)}(\omega^{-6M}E)
\end{aligned}
$$

with

$$
C^{(1,i)}(E) = C^{(1)}(E,\alpha_i)\,, \quad D^{(1,i)}(E) = D_0(E,\alpha_i)\,, \quad D^{(2,i)} \sim W[y,y_1]
$$

Third-order ODE

- Three sets (one for each value of α) of Bethe Ansatz-type equations for the eigenvalues E

$\longrightarrow$ Map to known quantum integrable model

- The third-order ODE is quasi-exactly solvable, for certain values of α (subset of eigenvalues and eigenfunctions can be constructed)

Massive ODE/IM Correspondence

Another direction of extension: include other Differential Equations and massive QFTs

- The correspondence concerned the mapping of certain ODEs to massless quantum field theories
- Lukyanov & Zamolodchikov showed how to include massive quantum field theories
- They had as a starting point the classical sinh-Gordon equation
- Here a correspondence between classical $A_{n-1}^{(1)}$ Toda field theories (classical integrable PDEs) and A_{n-1} Bethe Ansatz systems will be presented [2]

[2] P. Adamopoulou and C. Dunning, *Bethe ansatz equations for the classical $A_n^{(1)}$ affine Toda field theories*, J.Phys.A **47** (2014), 205205

Toda equations

- Describe the motion of n particles on a closed chain with exponential interactions [3]

$$\frac{d^2\eta_i}{dt^2} = 2\mathrm{e}^{2\eta_{i+1}-2\eta_i} - 2\mathrm{e}^{\eta_i-2\eta_{i-1}}\,, \quad i = 1,\ldots,n$$

  where η_i is the displacement of the i-th particle from its equilibrium position, with $\eta_{n+1} = \eta_1$

- A two-dimensional continuous generalisation of this model reads [4]

$$\partial_t^2\eta_i - \partial_x^2\eta_i = 2\mathrm{e}^{2\eta_{i+1}-2\eta_i} - 2\mathrm{e}^{\eta_i-2\eta_{i-1}}\,, \quad i = 1,\ldots,n$$

  with $\eta_i \equiv \eta_i(x,t) \in \mathbb{R}$ and $x\,,t \in \mathbb{R}$

[3] M. Toda, Vibration of a Chain with Nonlinear Interaction, Journal of the Physical Society of Japan 22 (1967), 431-436

[4] A. V. Mikhailov, Integrability of the two-dimensional generalization of toda chain, JETP Letters 30 (1979), 414-418

$A_{n-1}^{(1)}$ Toda field equations

- The two-dimensional $A_{n-1}^{(1)}$ Toda field theories are described by the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{n}(\partial_t \eta_i)^2 - (\partial_x \eta_i)^2 - \sum_{i=1}^{n}\exp(2\eta_{i+1} - 2\eta_i)$$

with $\eta_i \equiv \eta_i(x,t)$, periodic boundary conditions, and $\sum_{i=1}^{n}\eta_i = 0$

- Using coordinates $w = x + t$ and $\bar{w} = x - t$, the corresponding equations of motion are

$$2\,\partial_{\bar{w}}\partial_w \eta_i = \exp(2\eta_i - 2\eta_{i-1}) - \exp(2\eta_{i+1} - 2\eta_i)\,, \quad i = 1, \ldots, n\,,$$

with $\eta_i \equiv \eta_i(w,\bar{w})$ and $(w,\bar{w}) \in \mathbb{C}^2$

$A_{n-1}^{(1)}$ Toda field equations

- We are interested in a particular class of real-valued solutions to the $A_{n-1}^{(1)}$ Toda equations
- To this end, we first apply asymptotic analysis in certain asymptotic limits to a reduction of the $A_{n-1}^{(1)}$ Toda equations

$A_{n-1}^{(1)}$ Toda field equations

Asymptotic Analysis

- The combination $w\bar{w}$ remains invariant under a scaling of the variables

  $\longrightarrow$ perform a *symmetry reduction*:

$$t = \sqrt{2w\bar{w}}\,, \quad \eta_i(w,\bar{w}) = y_i(t)$$

- The $A_{n-1}^{(1)}$ Toda equations become a system of n coupled nonlinear ordinary differential equations

$$\frac{d^2}{dt^2}\,y_i + \frac{1}{t}\,\frac{d}{dt}\,y_i + \mathrm{e}^{2y_{i+1}-2y_i} - \mathrm{e}^{2y_i-2y_{i-1}} = 0\,, \quad i = 1,\ldots,n\,,$$

  which is of Painlevé III-form

$A_{n-1}^{(1)}$ Toda field equations

Asymptotic Analysis

- Asymptotic analysis to the system of equations provides the following leading order behaviours for $y_i(t)$

  - As $t \to 0$

    $$y_i(t) \sim (n - i - g_{n-i}) \ln t + b_i + \text{power series in t}$$

    with
    - $g_i, b_i \in \mathbb{R}$
    - $\sum_{i=0}^{n-1} g_i = n(n-1)/2$
    - $g_0 < g_1 < \cdots < g_{n-1}$ and $g_0 + n > g_{n-1}$

  - As $t \to \infty$

    $$y_i(t) = O(1)$$

$\to$ The constants g_i: related to certain parameters which enter (the ODEs) of the ODE/IM correspondence

$\to$ The asymptotic analysis provides for free certain relations which were imposed to the g_i

Lax pair $A^{(1)}_{n-1}$

- The $A^{(1)}_{n-1}$ Toda field equations are integrable

  The associated linear problem (Lax pair) is

$$\left(\partial_w + \widehat{U}(w,\bar{w},\lambda)\right)\mathbf{\Phi} = 0\,, \quad \left(\partial_{\bar{w}} + \widehat{V}(w,\bar{w},\lambda)\right)\mathbf{\Phi} = 0$$

  - U, V are functions from $(z,\bar{z}) \in \mathbb{C}^2$ to $A^{(1)}_{n-1}[\lambda,\lambda^{-1}]$
  - $\lambda = \mathrm{e}^{\theta} \in \mathbb{C}$ is the spectral parameter
    we have that

$$\widehat{U}(w,\bar{w},\lambda) = \partial_w \eta_i\, \delta_{ij} + \lambda\, C\,, \quad \widehat{V}(w,\bar{w},\lambda) = -\partial_{\bar{w}}\, \eta_i\, \delta_{ij} + \lambda^{-1} C\,,$$

$$(C)_{ij} = \exp(\eta_{j+1} - \eta_j)\, \delta_{i-1,j} \quad j = 1,\ldots,n$$

- The compatibility condition of the linear system of equations

$$\partial_w \widehat{V} - \partial_{\bar{w}} \widehat{U} + [\widehat{U},\widehat{V}] = 0$$

  (zero-curvature condition) is equivalent to the $A^{(1)}_{n-1}$ Toda field equations

How can we make a connection to quantum integrability?

Modified $A_{n-1}^{(1)}$ Toda field equations

$\longrightarrow$ We consider a modified version of the $A_{n-1}^{(1)}$ Toda equations

- Change of variables: $(w, \bar{w}) \to (z, \bar{z})$

$$\mathrm{d}w = p(z)^{1/n}\mathrm{d}z\,, \quad \mathrm{d}\bar{w} = p(\bar{z})^{1/n}\mathrm{d}\bar{z}\,,$$

with

$$p(t) = t^{nM} - s^{nM} \quad \text{and} \quad M, s \in \mathbb{R}_+$$

- Transformation of the fields

$$\eta_i(z, \bar{z}) \to \eta_i(z, \bar{z}) - \tfrac{1}{4n}(2i - n - 1)\ln(p(z)p(\bar{z}))$$

Modified $A_{n-1}^{(1)}$ Toda field equations

- The $A_{n-1}^{(1)}$ Toda equations transform to the modified $A_{n-1}^{(1)}$ Toda equations

$$2\partial_{\bar{z}}\partial_z \eta_i = \mathrm{e}^{2\eta_i - 2\eta_{i-1}} - \mathrm{e}^{2\eta_{i+1} - 2\eta_i} \quad \text{for} \quad i = 2, \ldots, n-1\,,$$

$$2\partial_{\bar{z}}\partial_z \eta_1 = p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1 - 2\eta_n} - \mathrm{e}^{2\eta_2 - 2\eta_1}\,,$$

$$2\partial_{\bar{z}}\partial_z \eta_n = \mathrm{e}^{2\eta_{n-1} - 2\eta_n} - p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1 - 2\eta_n}\,.$$

with $\eta_i \equiv \eta_i(z, \bar{z})$ and $(z, \bar{z}) \in \mathbb{C}^2$

Modified $A_{n-1}^{(1)}$ Toda field equations
Solutions

- A particular family of asymptotic solutions to the *modified* $A_{n-1}^{(1)}$ equations is obtained form the asymptotic solutions to the *original* $A_{n-1}^{(1)}$ equations:
  - As $z\bar{z} \to 0$

$$\eta_i(z,\bar{z}) \sim \tfrac{1}{2}(n-i-g_{n-i})\ln(z\bar{z}) + b_i + \sum_{k=1}^{\infty}\gamma_{ik}\left(z^{nkM} + \bar{z}^{nkM}\right) + \text{p.s. in } z\bar{z}$$

  - As $z\bar{z} \to \infty$

$$\eta_i(z,\bar{z}) = \tfrac{1}{4}(2i-n-1)\,M\ln(z\bar{z}) + o(1)$$

Lax Pair Modified $A_{n-1}^{(1)}$

- The Lax pair for the $A_{n-1}^{(1)}$ Toda equations is associated to that for the *modified* $A_{n-1}^{(1)}$ Toda equations by a gauge transformation

$$\left.\begin{array}{l}\big(\partial_w + \widehat{U}(w,\bar{w},\lambda)\big)\mathbf{\Phi} = 0 \\ \big(\partial_{\bar{w}} + \widehat{V}(w,\bar{w},\lambda)\big)\mathbf{\Phi} = 0\end{array}\right\} \xrightarrow[transform\,\eta_i]{(w,\bar{w})\to(z,\bar{z})} \left.\begin{array}{l}\big(\partial_z + \tilde{U}(z,\bar{z},\lambda)\big)\mathbf{\Phi} = 0 \\ \big(\partial_{\bar{z}} + \tilde{V}(z,\bar{z},\lambda)\big)\mathbf{\Phi} = 0\end{array}\right\} \xrightarrow[transf.]{gauge}$$

$$\big(\partial_z + U(z,\bar{z},\lambda)\big)\mathbf{\Psi} = 0\,, \quad \big(\partial_{\bar{z}} + V(z,\bar{z},\lambda)\big)\mathbf{\Psi} = 0\,,$$

with

$$A(z,\bar{z},\lambda) = g^{-1}g_z + g^{-1}\tilde{A}(z,\bar{z},\lambda)g\,, \qquad \mathbf{\Phi} = g\,\mathbf{\Psi}$$

and

$$(g)_{ij} = \left(\frac{p(\bar{z})}{p(z)}\right)^{n-\frac{2i-1}{4n}} \delta_{ij}$$

Lax Pair Modified $A_{n-1}^{(1)}$

Symmetries

- It is convenient to introduce $z = \rho\, \mathrm{e}^{i\phi}$, $\bar{z} = \rho\, \mathrm{e}^{-i\phi}$ with $(\rho, \phi) \in \mathbb{R}^2$
- We define the transformations
  - $\widehat{\Omega}: \quad \phi \to \phi + \frac{2\pi}{nM}\,, \quad \theta \to \theta - \frac{2\pi i}{nM}$
  - $\widehat{S}: \quad A(\lambda) \to S\, A\left(\sigma^{-1}\lambda\right) S^{-1} \quad \text{or} \quad A(\theta) \to S\, A(\theta - \frac{2\pi i}{n})\, S^{-1}$

    for any $n \times n$ matrix $A(\lambda)$, with $\sigma = \exp\left(\frac{2\pi i}{n}\right)$ and $(S)_{ij} = \sigma^i\, \delta_{i,j}$

Such groups of transformations are known as *reduction groups* [5]

[5] A. V. Mikhailov, The reduction problem and the inverse scattering method, Physica D Nonlinear Phenomena 3 (1981), 73-117

Lax Pair Modified $A_{n-1}^{(1)}$

Symmetries

- The matrices U, V of the linear problem are invariant under the action of the transformations $\widehat{\Omega}$, $\widehat{S}$

$$\widehat{\Omega}\left(U(\rho,\phi,\theta)\right) = U(\rho,\phi,\theta)\,, \quad \widehat{\Omega}\left(V(\rho,\phi,\theta)\right) = V(\rho,\phi,\theta)$$

$$\widehat{S}\left(U(\rho,\phi,\theta)\right) = U(\rho,\phi,\theta)\,, \quad \widehat{S}\left(V(\rho,\phi,\theta)\right) = V(\rho,\phi,\theta)$$

- The symmetries of U, V affect the auxiliary solution $\mathbf{\Psi}$ of the Lax pair

Lax Pair Modified $A_{n-1}^{(1)}$

Solution

- Consider a vector $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_n)^{\mathrm{T}}$
- A general solution to the modified $A_{n-1}^{(1)}$ linear problem reads

$$\Psi_i(z, \bar{z}, \lambda) = \begin{cases} -\lambda^{-1} \mathrm{e}^{\eta_i - \eta_{i+1}} \left(\partial_z \Psi_{i+1} + \partial_z \eta_{i+1} \Psi_{i+1} \right) & \text{for} \quad i = n-1, \ldots, 1 \\ \mathrm{e}^{\eta_n} \psi & \text{for} \quad i = n \end{cases}$$

- Where $\psi \equiv \psi(z, \bar{z}, \lambda)$ satisfies the nth-order differential equation

$$\Big((-1)^{n+1} D_n(\eta) + \lambda^n p(z) \Big) \psi = 0$$

with $D_n(\eta)$ the nth-order operator

$$D_n(\eta) = (\partial_z + 2\, \partial_z \eta_1)\, (\partial_z + 2\, \partial_z \eta_2) \ldots (\partial_z + 2\, \partial_z \eta_n)$$

Lax Pair Modified $A_{n-1}^{(1)}$

Solution

$\longrightarrow$ Finding an asymptotic solution for ψ allows for a solution $\mathbf{\Psi}$ to the Lax pair to be determined

$\longrightarrow$ Different asymptotic solutions η_i provide different potentials to the ODE for ψ

Lax Pair Modified $A_{n-1}^{(1)}$

Solution

- As $z\bar{z} \to 0$: solutions to the ODE for ψ behave as z^{g_j}, giving rise to n solutions to the Lax pair

$$\mathbf{\Xi}_j \sim \left(0, \ldots, 0, \mathrm{e}^{(g_j - j)(\theta + i\phi)}, 0 \ldots, 0\right)^{\mathrm{T}}$$

  where the $(n-j)^{\mathrm{th}}$ component is non-zero

- As $z\bar{z} \to \infty$: a WKB analysis of the ODE provides a solution ψ which leads to the asymptotic solution to the Lax pair

$$\mathbf{\Psi}(\rho, \phi, \theta, \mathbf{g}) \sim (\Psi_1, \ldots, \Psi_n)^{\mathrm{T}} \exp\left(-2\,\frac{\rho^{M+1}}{M+1}\cosh(\theta + i\phi(M+1))\right)$$

  where $\Psi_j = \exp\left(i\phi M(n - (2j-1))/2\right)$, $\mathbf{g} = \{g_0, \ldots, g_{n-1}\}$

Expand $\mathbf{\Psi}$ in the basis of solutions $\mathbf{\Xi}_j$

$$\mathbf{\Psi}(\rho, \phi, \theta, \mathbf{g}) = \sum_{j=0}^{n-1} Q_j(\theta, \mathbf{g})\, \mathbf{\Xi}_j(\rho, \phi, \theta, \mathbf{g})$$

Q-functions

The Q_j-functions are characterised by certain properties:

- they are expressed in terms of determinants of the solutions $\mathbf{\Xi}_j$, $\mathbf{\Psi}$
- they are quasi-periodic functions of θ:

$$Q_j(\theta, \mathbf{g}) = \exp\left(-\tfrac{2\pi i}{n}(g_j - 1)\right) Q_j\left(\theta - \tfrac{2\pi i}{n}\tfrac{(M+1)}{M}, \mathbf{g}\right)$$

  which follows from the symmetries $\widehat{\Omega}$, $\widehat{S}$ of the Lax pair

- they satisfy certain functional relations from which the A_{n-1} Bethe ansatz equations arise

Massless limit

- Consider the limit:

  Take $\bar{z} \to 0$ with z finite and small

  Then set $x = z\,\mathrm{e}^{\frac{\theta}{M+1}}$, $E = s^{nM}\mathrm{e}^{\frac{nM\theta}{M+1}}$ and take the limit $z \sim s \to 0$, $\theta \to \infty$ in the ODE for ψ

  This way the nth-order ODE appearing in the massless A_{n-1} ODE/IM correspondence is recovered

- In this limit the coefficients Q_i coincide with those of the massless quantum field theory related to the A_{n-1} Lie algebra

ODE/IM correspondence

$$\text{linear ODEs} \xrightarrow{\text{ODE/IM correspondence}} \text{BAE (massless QFTs)}$$

$$\text{nonlinear integrable PDEs} \xrightarrow{\text{ODE/IM correspondence}} \text{BAE (massive QFTs)}$$

Current Work

- Systems of nonlinear ODEs (second order) have emerged in the analysis of the classical $A_{n-1}^{(1)}$ Toda equations
- It is conjectured (ARS conjecture[6]) that the above ODEs possess very interesting analytic properties (Painlevé property)
- Confirm this conjecture and examine the integrability properties (Lax pair, Bäcklund transformations) (reduction to PIII)

Also

- Study analytic properties of Q-functions: asymptotic behaviour, pattern of zeros

[6] Ablowitz, M. J., Ramani, A, Segur, A., *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Lett. Nuovo Cimento **23** (1978), 333-338

Thank you for your attention

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